Tensor Contractions with Extended BLAS Kernels on CPU and GPU

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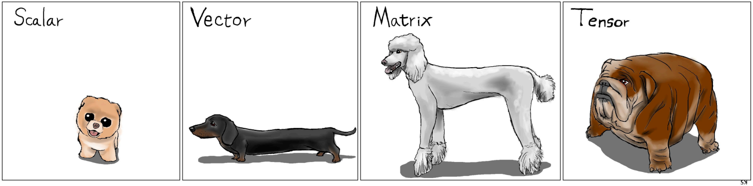
Tensor Contraction-Motivation



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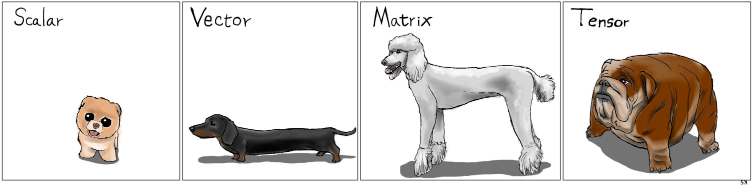

Modern data is inherently multi-dimensional

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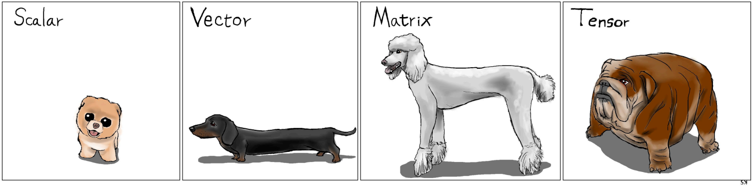

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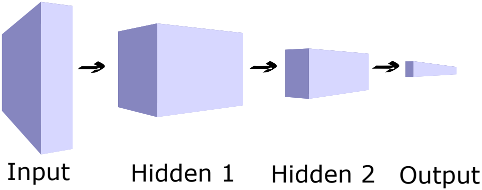

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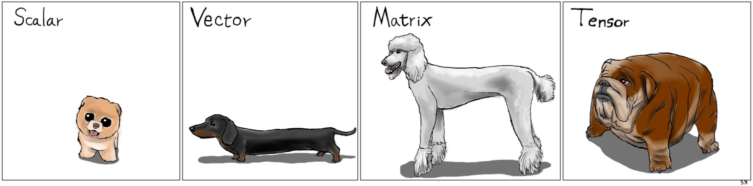

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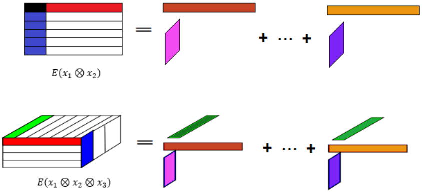

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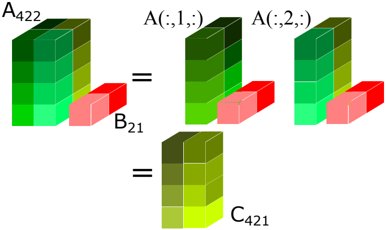

$$\text{e.g.} \quad C_{mnp} = A_{mnk} \, B_{kp}$$

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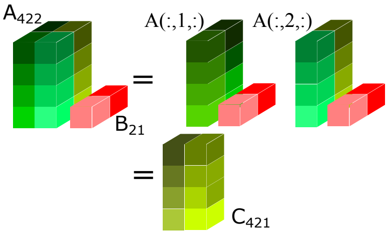

$$\text{e.g.} \quad C_{mnp} = A_{mnk}\, B_{kp}$$

Why do we need tensor contraction?

1. Core primitive of multilinear algebra.
2. BLAS Level 3: Unbounded compute intensity.

Tensor Contraction – Motivation

Lots of hot applications at the moment:

- Machine learning
- Deep learning
- e.g. Learning latent variable model with tensor decomposition:

Topic model [1]

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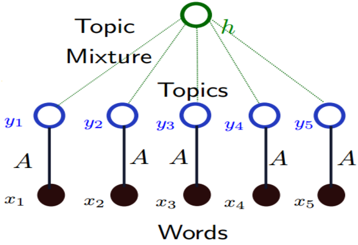
h: PDF of topics in a document.
A: Topic-word matrix.
$A_{ij} = \mathcal{P}(x_m = i | y_m = j)$



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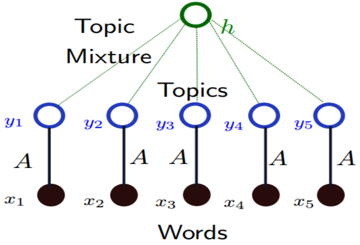

Form third-order tensor $M_3 = \mathbb{E}(x \otimes x \otimes x) = \sum_i h_i a_i \otimes a_i \otimes a_i$

[1] Tensor Decompositions for Learning Latent Variable Models, Anima Anandkumar, Rong Ge, Daniel Hsu et. al.

Tensor Contraction – Motivation

Distributed FFT

Tensor Contraction – Motivation

Distributed FFT

$$\begin{aligned}
T_{pib} &= S2T^{(p)}_{ijs}\, S_{pj(b+s)} && \Longrightarrow && T_{pib} = S2T^{(p)}_{i(js)}\, S_{p(js)b} \\
M_{pqb} &= S2M_{qi}\, S_{pib} && \Longrightarrow && M_{pq[b]} = S_{pi[b]}\, S2M^{T}_{qi} \\
M_{pqb'} &= M2M^{-}_{qm}\, M_{pmb^{-}} + M2M^{+}_{qm}\, M_{pmb^{+}} && \Longrightarrow && M_{pq[b']} = M_{pM[b]}\, M2M^{T}_{qM} \\
r_p &= 1_{ib}\, S_{pib} = 1_{qb}\, M_{pqb} && \Longrightarrow && r_p = 1_{(qb)}\, M_{p(qb)} \\
L_{pnb} &= M2L^{(p)}_{nms}\, M_{pm(b+s)} && \Longrightarrow && L_{pnb} = M2L^{(p)}_{n(ms)}\, M_{p(ms)b} \\
L_{pqb^{\pm}} &= L2L^{\pm}_{qm}\, L_{pmb'} && \Longrightarrow && L_{pq[b]} = L_{pM[b']}\, M2M_{qM} \\
T_{pib} &= L2T_{iq}\, L_{pqb} && \Longrightarrow && T_{pi[b]} = L_{pq[b]}\, S2M_{qi}
\end{aligned}$$

What do we have?

Tensor Contraction-Motivation

What do we have?

Tensor computation libraries

1. Arbitrary/restricted tensor operation of any order and dimension
    1. Tensortoolbox (Matlab)
    2. FTensor (C++)
    3. Cyclops (C++)
    4. BTAS (C++)
    5. All the Python...

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Efficient computing frame

1. Static analysis solutions
    1. PPCG [ISL] (polyhedral)
    2. TCE (DSL)
2. Parallel and distributed primitives
    1. BLAS, cuBLAS
    2. BLIS, BLASX, cuBLASXT

Tensor Contraction-Motivation

Libraries

Explicit permutation dominates.

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Explicit permutation dominates.

Consider $C_{mnp} = A_{km} B_{pkn}$.

1. $A_{km} \rightarrow A_{mk}$
2. $B_{pkn} \rightarrow B_{kpn}$
3. $C_{mnp} \rightarrow C_{mpn}$
4. $\boxed{C_{m(pn)} = A_{mk} B_{k(pn)}}$
5. $C_{mpn} \rightarrow C_{mnp}$

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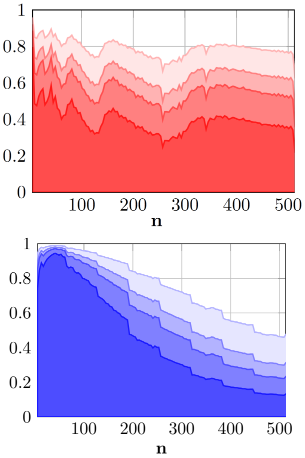


(Top) CPU. (Bottom) GPU. The fraction of time spent in copies/transpositions. Lines are shown with 1, 2, 3, and 6 transpositions.

Existing Primitives

GEMM

- Suboptimal for many small matrices.

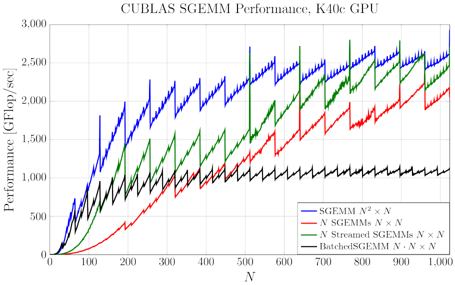
Pointer-to-Pointer BatchedGEMM

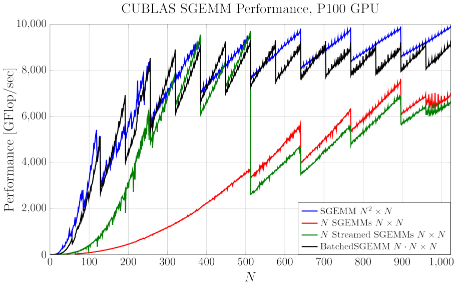
- Available in MKL 11.3β and cuBLAS 4.1

$$C[p] = \alpha \, \mathrm{op}(A[p]) \, \mathrm{op}(B[p]) + \beta \, C[p]$$

```
cublas<T>gemmBatched(cublasHandle_t handle,
                     cublasOperation_t transA, cublasOperation_t transB,
                     int M, int N, int K,
                     const T* alpha,
                     const T** A, int ldA,
                     const T** B, int ldB,
                     const T* beta,
                     T** C, int ldC,
                     int batchCount)
```

Pointer-to-Pointer BatchedGEMM

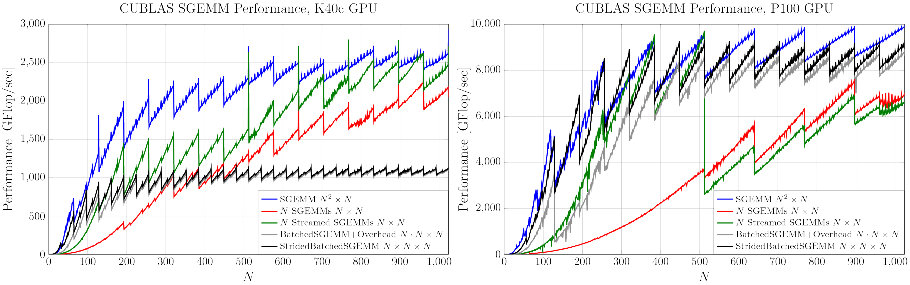




Existing Primitives

Pointer-to-Pointer BatchedGEMM

Except actually...



Solution: StridedBatchedGEMM

StridedBatchedGEMM

Exists!

... ~~Still no documentation?!?~~

Documentation as of last Tuesday!

StridedBatchedGEMM

Exists!

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In cuBLAS 8.0:

```
$$ grep StridedBatched -A 17 /usr/local/cuda/include/cublas_api.h
2320:CUBLASAPI cublasStatus_t cublasSgemmStridedBatched (cublasHandle_t handle,
2321-                                                       cublasOperation_t transa,
2322-                                                       cublasOperation_t transb,
2323-                                                       int m,
2324-                                                       int n,
2325-                                                       int k,
2326-                                                       const float *alpha,  // host or device pointer
2327-                                                       const float *A,
2328-                                                       int lda,
2329-                                                       long long int strideA,   // purposely signed
2330-                                                       const float *B,
2331-                                                       int ldb,
2332-                                                       long long int strideB,
2333-                                                       const float *beta,   // host or device pointer
2334-                                                       float *C,
2335-                                                       int ldc,
2336-                                                       long long int strideC,
2337-                                                       int batchCount);
...
```

StridedBatchedGEMM

```
cublas<T>gemmStridedBatched(cublasHandle_t handle,
                            cublasOperation_t transA, cublasOperation_t transB,
                            int M, int N, int K,
                            const T* alpha,
                            const T* A, int ldA1, int strideA,
                            const T* B, int ldB1, int strideB,
                            const T* beta,
                            T* C, int ldC1, int strideC,
                            int batchCount)
```

- Common use case for Pointer-to-pointer BatchedGEMM.
- No Pointer-to-pointer data structure or overhead.
- Performance on par with pure GEMM (P100 and beyond).

Tensor Contraction with Extended BLAS Primitives

$$C_{mnp} = A_{**} \times B_{***}$$

$$C_{mnp} \equiv C[m + n \cdot \text{ldC1} + p \cdot \text{ldC2}]$$

Case	Contraction	Kernel1	Kernel2	Case	Contraction	Kernel1	Kernel2
1.1	$A_{mk}B_{knp}$	$C_{m(np)} = A_{mk}B_{k(np)}$	$C_{mn[p]} = A_{mk}B_{kn[p]}$	4.1	$A_{kn}B_{kmp}$	$C_{mn[p]} = B^\top_{km[p]}A_{kn}$	
1.2	$A_{mk}B_{kpn}$	$C_{mn[p]} = A_{mk}B_{k[p]n}$	$C_{m[n]p} = A_{mk}B_{kp[n]}$	4.2	$A_{kn}B_{kpm}$	$C_{mn[p]} = B^\top_{k[p]m}A_{kn}$	
1.3	$A_{mk}B_{nkp}$	$C_{mn[p]} = A_{mk}B^\top_{nk[p]}$		4.3	$A_{kn}B_{mkp}$	$C_{mn[p]} = B_{mk[p]}A_{kn}$	
1.4	$A_{mk}B_{pkn}$	$C_{m[n]p} = A_{mk}B^\top_{pk[n]}$		4.4	$A_{kn}B_{pkm}$		
1.5	$A_{mk}B_{npk}$	$C_{m(np)} = A_{mk}B^\top_{(np)k}$	$C_{mn[p]} = A_{mk}B^\top_{n[p]k}$	4.5	$A_{kn}B_{mpk}$	$C_{mn[p]} = B_{m[p]k}A_{kn}$	
1.6	$A_{mk}B_{pnk}$	$C_{m[n]p} = A_{mk}B^\top_{p[n]k}$		4.6	$A_{kn}B_{pmk}$		
2.1	$A_{km}B_{knp}$	$C_{m(np)} = A^\top_{km}B_{k(np)}$	$C_{mn[p]} = A^\top_{km}B_{kn[p]}$	5.1	$A_{pk}B_{kmn}$	$C_{(mn)p} = B^\top_{k(mn)}A^\top_{pk}$	$C_{m[n]p} = B^\top_{km[n]}A^\top_{pk}$
2.2	$A_{km}B_{kpn}$	$C_{mn[p]} = A^\top_{km}B_{k[p]n}$	$C_{m[n]p} = A^\top_{km}B_{kp[n]}$	5.2	$A_{pk}B_{knm}$	$C_{m[n]p} = B^\top_{k[n]m}A^\top_{pk}$	
2.3	$A_{km}B_{nkp}$	$C_{mn[p]} = A^\top_{km}B^\top_{nk[p]}$		5.3	$A_{pk}B_{mkn}$	$C_{m[n]p} = B_{mk[n]}A^\top_{pk}$	
2.4	$A_{km}B_{pkn}$	$C_{m[n]p} = A^\top_{km}B^\top_{pk[n]}$		5.4	$A_{pk}B_{nkm}$		
2.5	$A_{km}B_{npk}$	$C_{m(np)} = A^\top_{km}B^\top_{(np)k}$	$C_{mn[p]} = A^\top_{km}B^\top_{n[p]k}$	5.5	$A_{pk}B_{mnk}$	$C_{(mn)p} = B_{(mn)k}A^\top_{pk}$	$C_{m[n]p} = B_{m[n]k}A^\top_{pk}$
2.6	$A_{km}B_{pnk}$	$C_{m[n]p} = A^\top_{km}B^\top_{p[n]k}$		5.6	$A_{pk}B_{nmk}$		
3.1	$A_{nk}B_{kmp}$	$C_{mn[p]} = B^\top_{km[p]}A^\top_{nk}$		6.1	$A_{kp}B_{kmn}$	$C_{(mn)p} = B^\top_{k(mn)}A_{kp}$	$C_{m[n]p} = B^\top_{km[n]}A_{kp}$
3.2	$A_{nk}B_{kpm}$	$C_{mn[p]} = B^\top_{k[p]m}A^\top_{nk}$		6.2	$A_{kp}B_{knm}$	$C_{m[n]p} = B^\top_{k[n]m}A_{kp}$	
3.3	$A_{nk}B_{mkp}$	$C_{mn[p]} = B_{mk[p]}A^\top_{nk}$		6.3	$A_{kp}B_{mkn}$	$C_{m[n]p} = B_{mk[n]}A_{kp}$	
3.4	$A_{nk}B_{pkm}$			6.4	$A_{kp}B_{nkm}$		
3.5	$A_{nk}B_{mpk}$	$C_{mn[p]} = B_{m[p]k}A^\top_{nk}$		6.5	$A_{kp}B_{mnk}$	$C_{(mn)p} = B_{(mn)k}A_{kp}$	$C_{m[n]p} = B_{m[n]k}A_{kp}$
3.6	$A_{nk}B_{pmk}$			6.6	$A_{kp}B_{nmk}$		

Tensor Contraction with Extended BLAS Primitives

Case	Contraction	Kernel1	Kernel2	Kernel3
1.1	$A_{mk}B_{knp}$	$C_{m(np)} = A_{mk}B_{k(np)}$	$C_{mn[p]} = A_{mk}B_{kn[p]}$	$C_{m[n]p} = A_{mk}B_{k[n]p}$
6.1	$A_{kp}B_{kmn}$	$C_{(mn)p} = B^{\top}_{k(mn)}A_{kp}$	$C_{m[n]p} = B^{\top}_{km[n]}A_{kp}$	

Example: Mappings to Level 3 BLAS routines

- Case 1.1, Kernel2: $C_{mn[p]} = A_{mk}B_{kn[p]}$

```
cublasDgemmStridedBatched(handle,
                          CUBLAS_OP_N, CUBLAS_OP_N,
                          M, N, K,
                          &alpha,
                          A, ldA1, 0,
                          B, ldB1, ldB2,
                          &beta,
                          C, ldC1, ldC2,
                          P)
```

Tensor Contraction with Extended BLAS Primitives

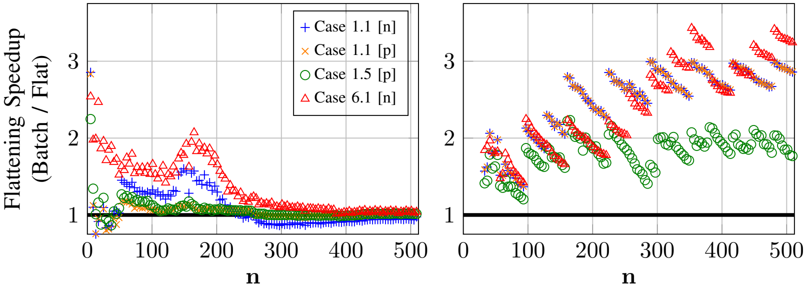
Case	Contraction	Kernel1	Kernel2	Kernel3
1.1	$A_{mk}B_{knp}$	$C_{m(np)} = A_{mk}B_{k(np)}$	$C_{mn[p]} = A_{mk}B_{kn[p]}$	$C_{m[n]p} = A_{mk}B_{k[n]p}$
6.1	$A_{kp}B_{kmn}$	$C_{(mn)p} = B^{\top}_{k(mn)}A_{kp}$	$C_{m[n]p} = B^{\top}_{km[n]}A_{kp}$	

Example: Mappings to Level 3 BLAS routines

- Case 6.1, Kernel2: $C_{m[n]p} = B^{\top}_{km[n]}A_{kp}$

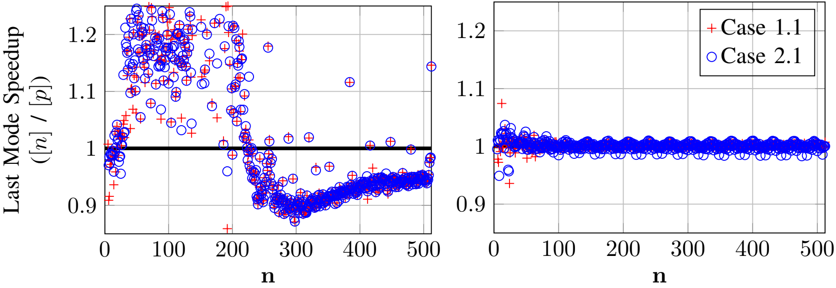
```
cublasDgemmStridedBatched(handle,
                          CUBLAS_OP_T, CUBLAS_OP_N,
                          M, P, K,
                          &alpha,
                          B, ldB1, ldB2,
                          A, ldA1, 0,
                          &beta,
                          C, ldC2, ldC1,
                          N)
```

Flatten V.S. SBGEMM



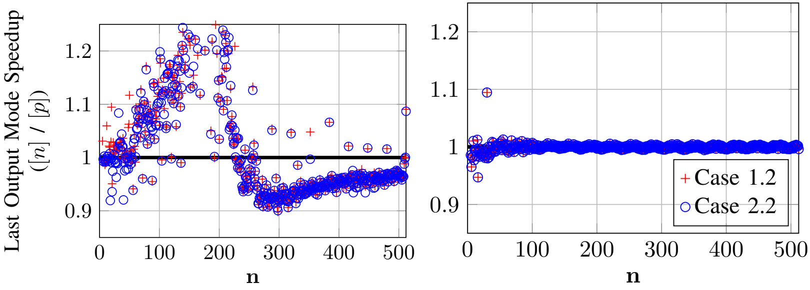
Prefer flattening to "pure" GEMM.

Batching in last mode versus middle mode



On CPU: Prefer batching in the last mode.

Mixed mode batching



On CPU: mode of the output tensor is more important than the batching mode of the input tensor.

Analysis

Exceptional Cases:
Cannot be computed by StridedBatchedGEMM.

Case	Contraction
3.4	$C_{mnp} = A_{nk}B_{pkm}$
6.4	$C_{mnp} = A_{kp}B_{nkm}$
	$C_{mnp} = A_{mkp}B_{mkn}$
	$C_{mnp} = A_{pkm}B_{nkp}$

Example of exceptional cases.

- These cases are precisely the interleaved GEMMs.
- When batching index is the major index in an argument:
  - That argument is interpreted as interleaved matrices.
  - May be one or both inputs and/or output.

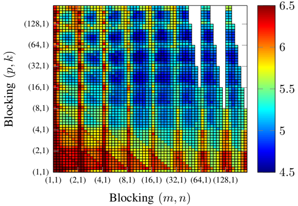
3D Tiled GEMM

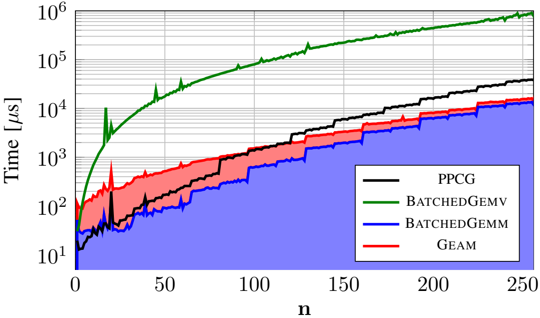
Implement GEMM with a 3D tile:

- Transpositions performed on the way to smem/reg.
- Keep canonical GEMM core.
- Considers three modes rather than two:
  - Major mode: $A_{\mathbf{m}nkpqr}$
  - Reduction mode: $A_{mn\mathbf{k}pqr}$
  - Aux (batch,row,col) mode: $A_{mnk\mathbf{p}qr}$ (Optional)
- Third tile dimension interpolates between pure GEMM and interleaved GEMM.
- Nested loop over remaining modes performs full contraction.

3D Tiled GEMM

Tilesize tuning with PPCG for exceptional cases:





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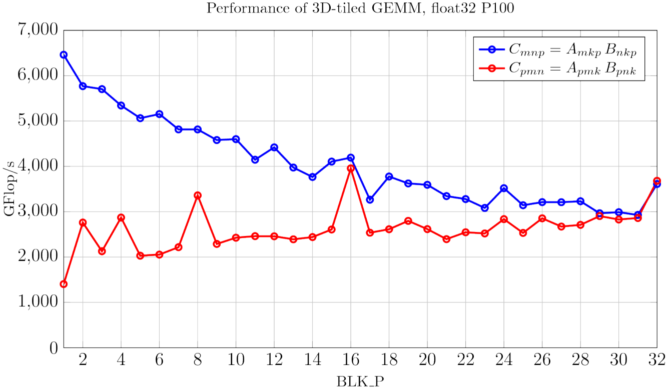


- $C_{mnp} = A_{mkp}\, B_{nkp}$: Increasing BLK_P decreases effective tile size.
- $C_{pmn} = A_{pmk}\, B_{pnk}$: Increasing BLK_P increases cache line utilization.
  - e.g. BLK_P$= 1, 2, 4, 8$
  - BLK_P $= 1$ equivalent to BLIS (strides in row and column)

3D Tiled GEMM – Interface?

Extend the StridedBatchedGEMM transpose parameters?

✓	C_{mnp}		$A_{pmk}\, B_{pkn}$	EX_N	EX_N
✓	C_{mpn}	$=$	$A_{pmk}\, B_{pnk}$	EX_N	EX_T
✗	C_{pmn}		$A_{pkm}\, B_{pkn}$	EX_T	EX_N
			$A_{pkm}\, B_{pnk}$	EX_T	EX_T

E.g. $C_{mn[p]} = A_{m[p]k}\, B_{[p]nk}$

```
cublasDgemmStridedBatched(handle,
                          CUBLAS_OP_N, CUBLAS_OP_EX_T,
                          M, N, K,
                          &alpha,
                          A, ldA2, ldA1,
                          B, ldB1, ldB2,
                          &beta,
                          C, ldC1, ldC2,
                          P);
```

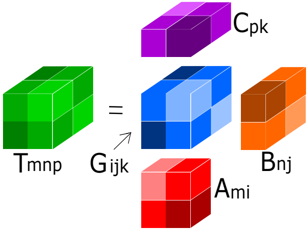
contract

```
contract(cublas::par,
         alpha,
         A, {M,P,K}, _<'m','p','k'>,
         B, {K,N,P}, _<'k','n','p'>,
         beta,
         C,          _<'m','n','p'>);
```

ROWIDX	COLIDX	BATIDX	REDIDX	Kernel	e.g.
0	0	0	0	mult	$C = A\, B$
0	0	0	1	dot	$C = A_k\, B_k$
0	0	1	0	XXX (scalar-mult)	$C_p = A_p\, B_p$
0	0	1	1	XXX (nested-dot?)	$C_p = A_{kp}\, B_{kp}$
0	1	0	0	scal	$C_n = A\, B_n$
0	1	0	1	gemv	$C_n = A_k\, B_{kn}$
0	1	1	0	dgmm (cublas?)	$C_{np} = A_p\, B_{np}$
0	1	1	1	batch_gemm (m=1?)	$C_{np} = A_{kp}\, B_{nkp}$
1	0	0	0	scal	$C_m = A_m\, B$
1	0	0	1	gemv	$C_m = A_{mk}\, B_k$
1	0	1	0	dgmm (cublas?)	$C_{mp} = A_{mp}\, B_p$
1	0	1	1	batch_gemm (n=1?)	$C_{mp} = A_{mkp}\, B_{kp}$
1	1	0	0	ger	$C_{mn} = A_m\, B_n$
1	1	0	1	gemm	$C_{mn} = A_{mk}\, B_{kn}$
1	1	1	0	batch_gemm (k=1?)	$C_{mnp} = A_{mp}\, B_{np}$
1	1	1	1	batch_gemm	$C_{mnp} = A_{mkp}\, B_{nkp}$

Applications: Tucker Decomposition

$$T_{mnp} = G_{ijk} A_{mi} B_{nj} C_{pk}$$



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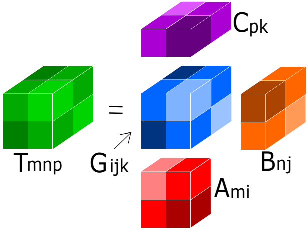
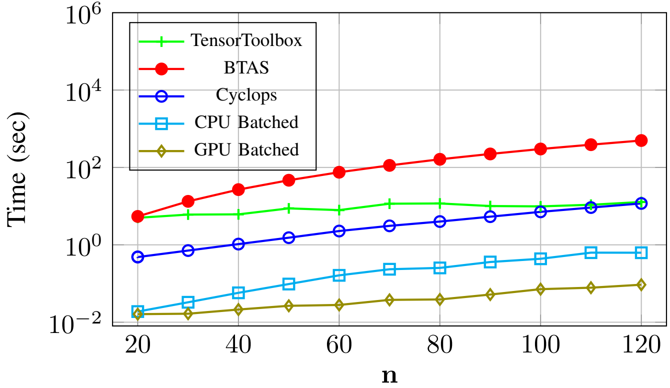


Main steps in the algorithm

- $Y_{mjk} = T_{mnp} B^t_{nj} C^t_{pk}$
- $Y_{ink} = T_{mnp} A^{t+1}_{mi} C^t_{pk}$
- $Y_{ijp} = T_{mnp} B^{t+1}_{nj} A^{t+1}_{mi}$

Applications: Tucker Decomposition

Performance on Tucker decomposition:



Applications: FFT

Low-Communication FFT for multiple GPUs.

- StridedBatchedGEMM composes 75%+ of the runtime.
  - Essential to the performance.
  - Two custom kernels are precisely interleaved GEMMs.

- 2 P100 GPUs: 1.3x over cuFFTXT.
- 8 P100 GPUs: 2.1x over cuFFTXT.

Conclusion

- StridedBatchedGEMM in cuBLAS for generalized tensor contractions.
- Avoid explicit transpositions or permutations.
- **10x**(GPU) and **2x**(CPU) speedup on small/moderate sized tensors.
- **Available in cuBLAS 8.0**

Conclusion

- StridedBatchedGEMM in cuBLAS for generalized tensor contractions.
- Avoid explicit transpositions or permutations.
- **10x**(GPU) and **2x**(CPU) speedup on small/moderate sized tensors.
- **Available in cuBLAS 8.0**
- Future work:
  - Exceptional case kernels/performance/interface??
  - Library Optimizations
    - Matrix stride zero – Persistent Matrix Strided Batched GEMM
    - Staged – RNNs: Staged Persistent Matrix Strided Batched GEMM

Thank you!
Questions?